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Stable Solutions to Homogeneous Difference-Differential Equations with Constant Coefficients

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Abstract:

In economic reality, reactions to external shocks often come with a delay. On the other hand, agents try to anticipate future developments. Both can lead to difference-differential equations with an advancing argument. These are more difficult to handle than either difference or differential equations, but they have the merit of added realism and increased credibility. We present a general method for determining the stability of any solution to a homogeneous linear difference-differential equation with constant coefficients and advancing arguments. We will also demonstrate the applicability of our concepts to economic modelling.

Key Words: Dynamic Systems, Stability, Monetary Transmission

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Analytical Instruments and an Application to Monetary Theory

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1. Introductory examples

Economically-motivated assumptions (see Krtscha [1993]) lead to the linear difference-differential equation with positive constants c and δ :

$$f'(x+\delta) = -c\cdot f(x)\,. \qquad (1.1)$$

The positive constant δ plays an important role in the nature of the solution. For $\delta = 0$ the solution to (1.1) would be uniquely determined by the condition $f(x_0) = y_0$, whereas for $\delta > 0$ the solution to (1.1) needs more: it is uniquely determined by any given function f: $[x_0, x_0+\delta] \to \mathbb{R}$ that is continuous and differentiable in $(x_0, x_0+\delta)$, implying the differentiability of f in (x_0, ∞). Then equation (1.1) leads to the formula

$$f(x+n\delta) = -c\int_0^x f(t+(n-1)\delta)\,dt + f(n\delta)\,,$$

by which we can regressively calculate f(x) for all $n \in \mathbb{N}$. We want to give a specific example:

$$f'(x+1) = -f(x) \text{ with } f(x) = 1-x\,,\ x \in [0, 1]\,. \qquad (1.2)$$

Then, by means of the regression-formula for $x \in [0, 1]$, we get

$$f(x+1) = -x + x^2/2\,,$$

implying $f(1) = 0$ and $f(2) = -1/2$, $f(x+2) = x^2/2 - x^3/3 - ½$, $f(x+3) = -x^3/3! + x^4/4! + x/2 - 1/3!$ and so on.

Now at least two questions arise:

- Is this special solution f(x) to (1.2) stable in the following sense: "f(x) converges if x converges against infinity"?

- Does every solution to the difference-differential equation (1.2) converge?

As to equation (1.2), it is almost obvious that every solution is stable, but with respect to equation (1.1), we will see later that the answer depends on the constant c. But before we do so, we will show the consequences of a slight modification:

$$f'(x+1) = f(x) \text{ with } f(x) = 1 + x\,,\ x \in [0, 1]\,, \tag{1.3}$$

that has the obviously non-stable solution

$$f(x) = \sum_{j=0}^{n} \frac{(x-j+1)^j}{j!},\ \ n \le x \le n+1 \ \text{ for } n = 1, 2, 3, ...$$

The question is now whether it is possible to generate a stable solution to this difference-differential equation by choosing another suitable *rational* initial function f: $[0, 1]\rightarrow \mathrm{IR}$. Although the answer is "no", there exist infinitely many *irrational* initial functions implying a stable solution.

In order to prove our answers, we have to revert to an old theorem by Hilb (1918) which is almost forgotten because it appeared to be too complicated for practical use. Fortunately, the theorem also has important implications which are useful for application.

2. A general method of investigating stability

Hilb (1918) proved a basic theorem, which we now present in a form adapted to our problem:

<u>Theorem 1.</u>

Let the general homogeneous linear difference-differential equation with the real constants k_{pq} and the monotonic increasing delays $h_0, h_1, h_2, ... ,h_n$

$$\sum_{p=0}^{m}\sum_{q=0}^{n} k_{pq} f^{(p)}(x+h_q) = 0,$$

where $h_0 = 0$, $f^{(0)}(x) = f(x)$, have the characteristic equation

$$\Pi(z)=\sum_{p=0}^{m}\sum_{q=0}^{n}k_{pq}z^{p}e^{h_q z}=0\,,$$

that comes from inserting $f(x) = e^{zx}$ in the difference-differential equation. (Moreover, let for the moment all the complex solutions z_ν to the characteristic equation be single.) Then one can choose any m-times differentiable function $f: (x_0,x_0+h_n)\to\mathbb{R}$ satisfying the difference-differential equation in $x = x_0 + h_n$, if the coefficients k_{m0} and k_{mn} are different from 0. However, if the latter condition is not satisfied, f must generally be infinitely often differentiable[1] in $x_0 + h_n$. By this choice, the solution f(x) to the difference-differential equation is uniquely determined for all $x > x_0$, and f(x) is given by the uniformly convergent series

$$f(x)=\sum_{\nu}\frac{-C_\nu(x_0)}{\Pi'(z_\nu)}\cdot e^{z_\nu x}\,,$$

with

$$C_\nu(x_0)=\sum_{p=0}^{m}\sum_{q=0}^{n-1}k_{pq}z_\nu{}^{p}e^{h_q z_\nu}\int_{x_0+h_q}^{x_0+h_n}e^{-z_\nu\mu}f(\mu)\,d\mu-\sum_{p=1}^{m}\ \sum_{q=0}^{n}k_{pq}z_\nu{}^{p}\sum_{t=0}^{p-1}\frac{f^{(t)}(x_0+h_q)e^{-z_\nu x_0}}{z_\nu{}^{t+1}}\quad,$$

where the convergence is uniform, i.e. it does not depend on the $x\in[x_0,K]$, for any big K.

Hilb arrived at this result by the use of the “Residuensatz” of Cauchy, and at the end of his 33-page paper he mentions that this is the reason why the formula for the coefficients C_ν holds even if the solutions to the characteristic equation are not unique. In this case, however, the C_ν are of a polynomial form in x, the degree being m-1, if z_ν is an m-fold solution to the characteristic equation. The latter fact is also mentioned by Hadeler (1974, p.170).

The consequence of this convergence is, roughly speaking, that in order to study the behaviour of the solution f(x) when x tends to infinity, we can stop summarising the above series at a sufficiently big ν. We can prove the following theorem:

[1] Krtscha (1991) gives an example of this case, but we could also give examples where it is not necessary that f be infinitely often differentiable.

Theorem 2.

Let M:= $\{z_\nu = u_\nu + iv_\nu$ with real $u_\nu, v_\nu\}$ be the set of all solutions to the characteristic equation $\Pi(z) = 0$ in theorem 1; let further the solution f(x) to the corresponding difference-differential equation be generated by some $z_{\nu j} = u_{\nu j} + iv_{\nu j} \in$ M, i.e. f(x) is a linear combination of the functions $e^{z_{\nu j}x}$, which are independent, being also proved by Hilb (1918). Then the solution f(x) is stable if and only if all $u_{\nu j}$ are negative.

Proof of Theorem 2:

Assuming there exists a positive $u_{\nu j}$, then it is obvious that |f(x)| tends to infinity if x tends to infinity, because all $e^{z_\nu x}$ are independent. On the other hand, if all $u_{\nu j}$ are negative, then watching the solution f(x) in $(x_0, K]$ by theorem 1 we may take only a finite linear combination f(x) of the

$$\frac{C_{\nu j}(x_0)}{\Pi'(z_{\nu j})} e^{z_{\nu j}x},$$

without making a big difference to the true solution f(x). Then this linear combination f(x) is stable because of the decreasing |f(x)| to zero; for all $C_{\nu j}(x_0)$ are constant or utmost polynomial in x.

Using theorem 2, we will now prove the assertions made in the introduction, and then we proceed to solve a problem from economic theory.

3. Continuation of the introductory examples

We insert a complex-valued function $f(x) = e^{zx}$ with z = a + ib, (a, b $\in$ IR), and $f'(x) = ze^{zx}$ in (1.1), and then we have to solve the characteristic equation

$$z \cdot e^z = -c. \qquad (3.1)$$

This equation has infinitely many complex solutions z_ν, and we now know that every differentiable solution to a linear functional differential equation with constant coefficients can be expressed by a convergent generalised Fourier series,

$$\sum_\nu \Gamma_\nu \cdot e^{z_\nu},$$

where the coefficients Γ_v are constants if all z_v are single solutions to (3.1). However, it is nearly impossible to calculate all z_v exactly, and Hilb's formulas for the Γ_v, moreover depending on the given function f in the starting interval $[x_0, x_0 + \delta]$, are too complicated for practical calculation. (Nevertheless, it is sometimes possible to decide, when solutions $z_v = a_v + ib_v$ with negative a_v exist, i.e. stable solutions to the differential equation with an advancing argument.)

Returning to the general solution to (3.1), we obtain.

$$(a + ib)\, e^{a+ib} = - c \, , \tag{3.2}$$

where b can be positive or negative. By splitting (3.2), we get the system

$$e^{a} (a \cos b - b \sin b) = - c \tag{3.3}$$

$$b \cos b + a \sin b = 0 \quad , \tag{3.4}$$

For $\sin b \neq 0$, equation (3.4) implies $a = -b \cos b / \sin b$. Inserting this term into (3.3), we get

$$b / \sin b = c \cdot e^{b \cos b / \sin b}. \tag{3.5}$$

The graph of the left-hand side $b/\sin b$ of (3.5) is intersected with the graph of the right-hand side of (3.5), and we see infinitely many solutions b_k for $b>\pi$, all implying negative a_k, but only one solution a_1 for $b_1 \in [0, \pi]$. This solution implies a negative a_1 for $c < \pi/2$ and further a positive a_1 for $c > \pi/2$. The case $\sin b_1 = 0$ implies $b_1 = 0$ and, by (3.3), a negative a_1; that means a stable real solution.

Hence, by means of theorem 2, we can state that every solution of (1.1) is stable if and only if $c \in (0, \pi/2)$. For $c = \pi/2$, we obtain $a_1 = 0$, $b_1 = \pi/2$, and the corresponding solution is non-stable, but periodic.

This example shows all possible types of solutions to a linear difference-differential equation with constant coefficients, and, because all solutions to the characteristic equation can be found graphically, it can be easily followed.

As to equation (1.3), there exist infinitely many $z_v = a_v + ib_v$ with negative a_v, but also one real positive z_v. If we want a stable solution, we have to take a linear combination of the $e^{z_v x}$ with the negative a_v, but this is not a *rational* function.

For the following economic model, which is developed by von Kalckreuth and Schröder (2002), it is again easy to find the real solutions z_v to the characteristic equation graphically. However, as global stability is a side condition of the model, we have to show that all other solutions $z_v = a_v + ib_v$ have positive a_v, implying not bounded solutions. If this can be achieved, there is a unique *stable* solution to the system that determines the model dynamics.

In the first examples, we did not show the economic background, which is already published, as we mainly wanted to give an overview of a general mathematical solving possibility that is not commonly used. In Krtscha (1991) and some other papers, the fixed point principle of Banach is applied. The disadvantage of this method – compared with the application of theorem 2 – is its limitation to local statements.

4. A model of monetary transmission

In order to investigate the interactions between the service life of capital, the term structure of interest rates and the impact of monetary policy on open economies, von Kalckreuth and Schröder (2002) develop a dynamic macroeconomic model. This model considers an interest-rate structure within the framework of the Dornbusch (1976) overshooting model. Whereas the central bank is able to influence the nominal short-term rate, aggregate demand depends on the real long-term rate. The interest-rate structure embodied in this model leads to an advancing argument in a system of functional equations. The authors solve this system by imposing additional restrictions, in effect boiling the dynamics down to a first-order differential equation.

Here, we want to investigate the *unrestricted* model dynamics. This leads to an interesting linear differential-difference equation with advancing argument. We show that there is a unique stable solution, which is identical to the solution described by von Kalckreuth and Schröder (2002) in solving the restricted model.

The unrestricted model contains the following equations:

$$\overline{M} - P(t) = \alpha_1 \overline{Y} - \alpha_2 i(t) \tag{4.1}$$

$$r(t) = i(t) - \dot{P}(t) \tag{4.2}$$

$$D(t) = \beta_0 + \beta_1 (E(t) - P(t)) + \beta_2 \overline{Y} - \beta_3 R_\Omega(t) \tag{4.3}$$

$$\dot{P}(t) = \Gamma\left(D(t) - \overline{Y}\right) \tag{4.4}$$

$$i(t) = i^* + \dot{E}(t) \tag{4.5}$$

$$\dot{R}_\Omega(t) = \frac{1}{\Omega}\left(r(t+\Omega) - r(t)\right) \tag{4.6}$$

A bar denotes a steady-state value. All coefficients in (4.1)–(4.6) are strictly positive. (4.1) is the Cagan-form money-market equation with *M*, *P* and *Y* as the logarithms of nominal money supply, price level and constant real income and *i* as the nominal money-market interest rate. While *P* is restricted to be a globally continuous function of time, all the other endogenous variables are allowed to jump discontinuously in response to an unexpected shock in $t = 0$. Equation (4.2) defines the real money-market interest rate *r* as the difference between the short-term nominal interest rate *i* and inflation. In (4.3), the logarithm of aggregate demand, *D,* depends on the logarithms of the real exchange rate $E - P$, the constant real income and the long-term real interest rate R_Ω, with maturity Ω. For simplicity, a uniform service life of capital is assumed. Eq. (4.4) is a simple Phillips relationship, in which the rate of inflation $\dot{P} = dP/dt$ is determined by the ratio of (variable) aggregate demand to (constant) supply. Equation (4.5) represents the open interest-parity condition, with i^* as the given nominal short-term interest rate in the international money market. Equation (4.6), finally, relates the change in the real long-term interest rate to the difference between future short-term rates and present short-term rates.

The last equation brings in an advancing argument in an otherwise linear system of differential equations. Because of its importance for our mathematical problem, the relationship between short and long-term rates will now be derived from first principles, as an arbitrage equation[2] for bonds of finite maturity .

Complete foresight in a perfect asset market implies the equality of the instantaneous real rates of return on bonds and investments in the money market. Consider a zero bond of arbitrary time to maturity Ω, issued at time t. Let N be the issue price and $R_\Omega(t)$ the long-term rate for bonds with time to maturity Ω. At time $t + \Omega$, then, the holder of the bond receives a payment of $N\exp(\Omega R_\Omega(t))$. Since there are no interest payments until maturity, according to the Hotelling rule, the arbitrage condition is given by:

[2] See, for example, McCulloch (1971). Fisher/Turnovsky (1992) also use this equation in the context of a dynamic macroeconomic model.

$$\frac{\dot{K}(s)}{K(s)} = r(s) \ , \tag{4.7}$$

with $K(s)$ as the real market value of a bond at any point in time s between t and $t+\Omega$. Taking into consideration the terminal condition that, at maturity, the real market value $K(s)$ is bound to be equal to the real value of principal and accrued interest, the general solution of (2.7) becomes:

$$K(s) = N \exp\left(\Omega R_\Omega(t) - \int_s^{t+\Omega} r(\tau) d\tau \right) . \tag{4.8}$$

At the date of issue, however, the value of this expression must be equal to *N*, which immediately yields the arbitrage equation:

$$R_\Omega(t) = \frac{1}{\Omega} \int_t^{t+\Omega} r_\tau d\tau \ . \tag{4.9}$$

Equation (4.9) gives us the term structure of interest rates, according to the expectation theory, for the case of continuous interest compounding. The (continuous) long-term rate R_Ω is determined as the arithmetic mean of the short-term rates within the relevant time interval. The former thus anticipates the movement of the latter. Taking the derivative on both sides yields equation (4.6). The interest rate R_Ω, by definition, gives us the cost of capital of an investment project characterised by one single payment in t and one single, certain return of V at the end of its lifetime Ω. Investigating the dynamic system for varying Ω thus permits us to describe the effects of a decreasing service life of capital, as a result of accelerated technical progress, on the dynamics of macroeconomic adjustment to various kinds of shocks.

In order to solve system (4.1)–(4.6) analytically, von Kalckreuth and Schröder (2002) use the following *additional* restriction on the time path:

$$\dot{P}_t = \lambda \tilde{P}_t , \qquad \lambda \in \mathbf{R},\ \lambda < 0 . \tag{4.10}$$

This restriction requires the perfect foresight paths to be of an especially simple, adaptive structure. Here, we do not intend to discuss the economic results of von Kalckreuth and Schröder (2002).[3] Instead, *we want to investigate whether it is pos-*

[3] The paper is available on the website of the co-author, www.von-kalckreuth.de

sible to drop the additional restriction without losing the results. This is possible, if there are no additional *stable* solutions to the system, apart from the solution found by imposing the restriction (4.10). Unstable solutions are ruled out by the assumption of perfect foresight, if we assume that all real (i.e. not nominal) variables are bounded.

First of all, we will reduce the system to one single dynamic equation. Long-run equilibrium is characterised by the conditions

$$\dot{P}(t)=0 \text{ and } \dot{E}(t)=0 . \tag{4.11}$$

Substituting these conditions into the system (4.1) – (4.6) readily yields a particular solution to the system, the steady-state solution. We can therefore concentrate on finding the solutions to the following set of homogeneous equations:[4]

$$P(t)=\alpha_2 i(t) \tag{4.1'}$$

$$r(t)=i(t)-\dot{P}(t) \tag{4.2'}$$

$$D(t)=\beta_1 E(t)-\beta_1 P(t)-\beta_3 R_\Omega(t) \tag{4.3'}$$

$$\dot{P}(t)=\Gamma D(t) \tag{4.4'}$$

$$i(t)=\dot{E}(t) \tag{4.5'}$$

$$\dot{R}_\Omega(t)=\frac{1}{\Omega}\left(r(t+\Omega)-r(t)\right) \tag{4.6'}$$

Repeated substitution yields the following homogeneous differential-difference equation with advancing argument:

$$\ddot{P}(t)=-AP(t+\Omega)+BP(t)+AC\dot{P}(t+\Omega)-BC\dot{P}(t), \tag{4.12}$$

where:

$$A=\frac{\Gamma\beta_3}{\Omega\alpha_2}>0 \tag{4.13}$$

$$B=\frac{\Gamma}{\alpha_2}(\beta_1+\frac{\beta_3}{\Omega})>A>0 \tag{4.14}$$

$$C=\alpha_2>0 . \tag{4.15}$$

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[4] The levels of the variables have to be interpreted as deviations from the steady state. In order to save notation, we will not introduce new symbols.

With

$$\Pi(z) := B - Ae^{\Omega z} - BCz + ACze^{\Omega z} - z^2 \;, \tag{4.16}$$

we get the characteristic equation

$$\Pi(z) = 0 \,. \tag{4.17}$$

It is not difficult to show graphically, as von Kalckreuth and Schröder (2002) have done, that there is one and only one negative real solution to the characteristic equation. But only if this stable solution is unique does the model have explanatory power. We will proceed to prove that this is indeed the case.

<u>Lemma :</u>

There is no complex solution $z = -u + iv$ with $u \geq 0$ and $v \neq 0$ to the characteristic equation $B - Ae^{\Omega z} - BCz + ACze^{\Omega z} - z^2 = 0$ with $0 < A < B$ and $0 \leq C$.

<u>Proof:</u>

By inserting $z = -u + iv$ in the characteristic equation and splitting up into the real and the imaginary part, we obtain a system of two equations:

$$\frac{B}{A} = e^{-u\Omega} \cos v\Omega + \frac{1}{A} \cdot \frac{(u^2 - v^2)(1 + uC) + 2uv^2 C}{(1 + uC)^2 + v^2 C^2} \tag{4.18}$$

$$\frac{1}{A} = e^{-u\Omega} \sin v\Omega \cdot \frac{(1 + uC)^2 + v^2 C^2}{(v^2 - u^2)vC + (1 + uC)2uv} \,. \tag{4.19}$$

Since $\dfrac{B}{A} > 1$ and $e^{-u\Omega} \cos v\Omega \leq 1$, equation (4.18) implies the inequality

$$(u^2 - v^2)(1 + uC) + 2uv^2 C > 0 \,; \tag{4.20}$$

so that $u = 0 \wedge v \neq 0$ is impossible. A purely imaginary solution to the characteristic equation, i.e. a periodic solution, can thus be excluded.

In order to consider solutions within the second and third quadrant of the complex plane, we define:

$v := mu$, with $u > 0$ and fixed real $m \neq 0$,

and insert it in (4.19), to get:

$$\frac{1}{A} = e^{-u\Omega} \frac{\sin mu\Omega}{mu^2} \cdot \frac{(1+uC)^2 + m^2u^2C^2}{(m^2+1)uC+2} \ .$$

Inserting this in (2.18), we obtain the equation:

$$\frac{B}{A} = e^{-\mu\Omega}\left[\cos mu\Omega + \frac{\sin mu\Omega}{m}\frac{\left(1-m^2\right)\left(1+uC\right)+2um^2C}{(m^2+1)uC+2}\right] \quad , \tag{4.21}$$

which implies the inequality:

$$\frac{B}{A} \le e^{-\mu\Omega}\left[1+\left|\frac{\sin mu\Omega}{m}\right| \cdot \left|\frac{1-m^2+uC\left(1+m^2\right)}{2+uC\left(1+m^2\right)}\right|\right] \ . \tag{4.22}$$

Owing to (4.20), the last ratio in (4.22) is positive, and as $1-m^2<2$, this expression is bounded by 1. Hence (4.22) implies:

$$\frac{B}{A} \le \frac{1+\left|\frac{\sin mu\Omega}{m}\right|}{e^{u\Omega}} \le \frac{1+\frac{|m|u\Omega}{|m|}}{1+u\Omega+\frac{u^2\Omega^2}{2!}+\ldots} \le 1 \ .$$

This last inequality is in contradiction to $A<B$. This simultaneously proves our lemma, and the fact that a non-exploding and non-trivial solution to the differential-difference equation can only be based upon a negative real solution of the characteristic equation to the differential difference equation. It is easily shown graphically that one and only one negative real solution $z_1 = \lambda*$ exists, see von Kalckreuth and Schröder (2002). So the series of the non-exploding solution is reduced to

$$f(x) = -\frac{e^{z_1x}}{\Pi'(z_1)}C_1,$$

where the constant C_1, defined in theorem 1, can be chosen in a way that the initial condition for $t=0$ is fulfilled.

This result confirms that dropping the additional restrictions in the basic model and thus making the generalisation into an unrestricted perfect foresight model does not lead to additional stable solutions for the dynamic system. Thus the economic conclusions developed in von Kalckreuth and Schröder (2002) are not affected by this generalisation.

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